



15.3 Check Your Understanding Find the voltages across the resistor, the capacitor, and the inductor in the circuit of **Figure 15.11** using $v(t) = V_0 \sin \omega t$ as the output of the ac generator.

15.4 | Power in an AC Circuit

Learning Objectives

By the end of the section, you will be able to:

- Describe how average power from an ac circuit can be written in terms of peak current and voltage and of rms current and voltage
- Determine the relationship between the phase angle of the current and voltage and the average power, known as the power factor

A circuit element dissipates or produces power according to $P = IV$, where I is the current through the element and V is the voltage across it. Since the current and the voltage both depend on time in an ac circuit, the instantaneous power $p(t) = i(t)v(t)$ is also time dependent. A plot of $p(t)$ for various circuit elements is shown in **Figure 15.16**. For a resistor, $i(t)$ and $v(t)$ are in phase and therefore always have the same sign (see **Figure 15.5**). For a capacitor or inductor, the relative signs of $i(t)$ and $v(t)$ vary over a cycle due to their phase differences (see **Figure 15.7** and **Figure 15.9**). Consequently, $p(t)$ is positive at some times and negative at others, indicating that capacitive and inductive elements produce power at some instants and absorb it at others.

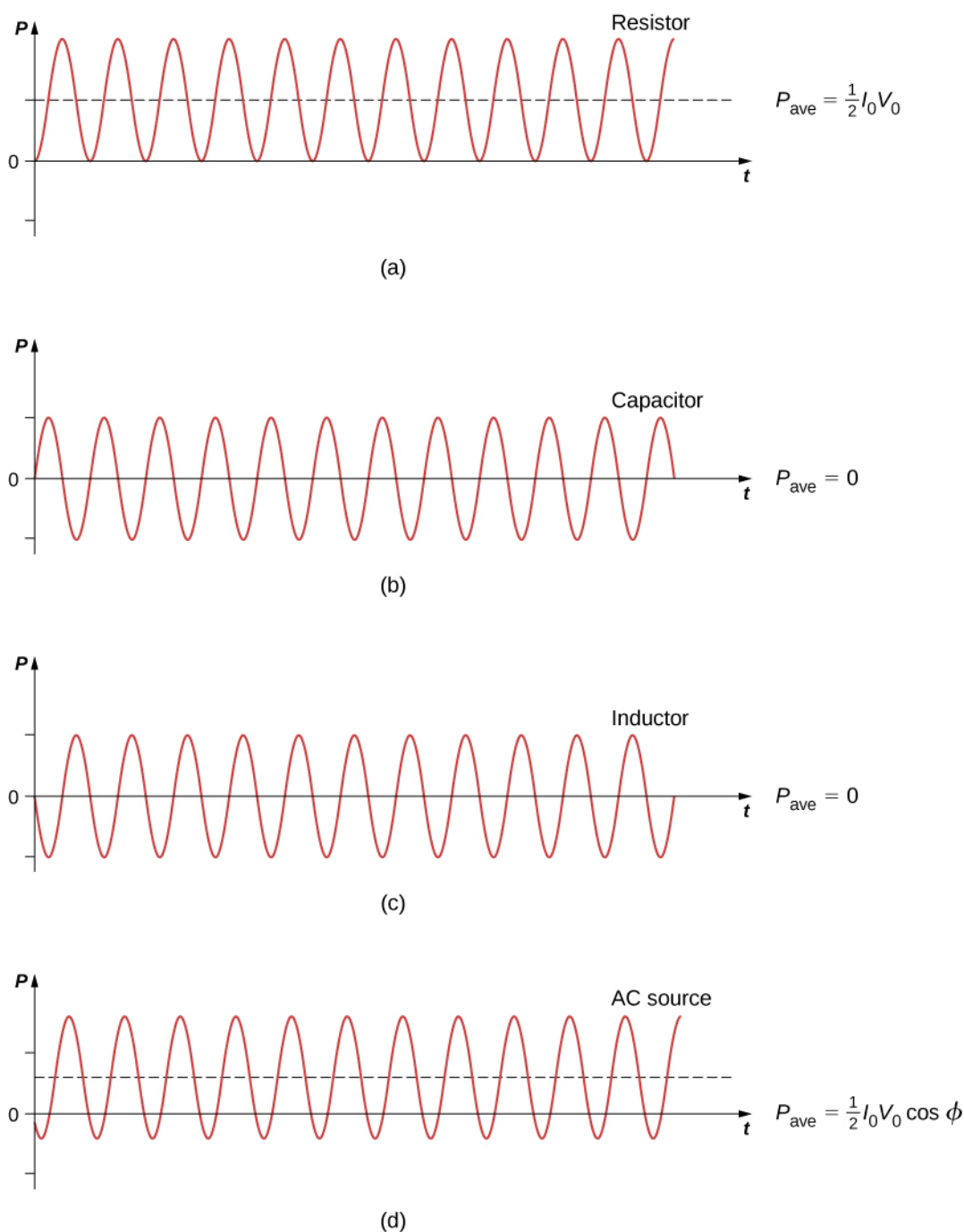


Figure 15.16 Graph of instantaneous power for various circuit elements. (a) For the resistor, $P_{\text{ave}} = I_0 V_0/2$, whereas for (b) the capacitor and (c) the inductor, $P_{\text{ave}} = 0$. (d) For the source, $P_{\text{ave}} = I_0 V_0(\cos \phi)/2$, which may be positive, negative, or zero, depending on ϕ .

Because instantaneous power varies in both magnitude and sign over a cycle, it seldom has any practical importance. What we're almost always concerned with is the power averaged over time, which we refer to as the **average power**. It is defined by the time average of the instantaneous power over one cycle:

$$P_{\text{ave}} = \frac{1}{T} \int_0^T p(t) dt,$$

where $T = 2\pi/\omega$ is the period of the oscillations. With the substitutions $v(t) = V_0 \sin \omega t$ and $i(t) = I_0 \sin(\omega t - \phi)$,

this integral becomes

$$P_{\text{ave}} = \frac{I_0 V_0}{T} \int_0^T \sin(\omega t - \phi) \sin \omega t \, dt.$$

Using the trigonometric relation $\sin(A - B) = \sin A \cos B - \sin B \cos A$, we obtain

$$P_{\text{ave}} = \frac{I_0 V_0 \cos \phi}{T} \int_0^T \sin \omega t \, dt - \frac{I_0 V_0 \sin \phi}{T} \int_0^T \sin^2 \omega t \cos \omega t \, dt.$$

Evaluation of these two integrals yields

$$\frac{1}{T} \int_0^T \sin^2 \omega t \, dt = \frac{1}{2}$$

and

$$\frac{1}{T} \int_0^T \sin \omega t \cos \omega t \, dt = 0.$$

Hence, the average power associated with a circuit element is given by

$$P_{\text{ave}} = \frac{1}{2} I_0 V_0 \cos \phi. \quad (15.12)$$

In engineering applications, $\cos \phi$ is known as the **power factor**, which is the amount by which the power delivered in the circuit is less than the theoretical maximum of the circuit due to voltage and current being out of phase. For a resistor, $\phi = 0$, so the average power dissipated is

$$P_{\text{ave}} = \frac{1}{2} I_0 V_0.$$

A comparison of $p(t)$ and P_{ave} is shown in **Figure 15.16(d)**. To make $P_{\text{ave}} = (1/2) I_0 V_0$ look like its dc counterpart, we use the rms values I_{rms} and V_{rms} of the current and the voltage. By definition, these are

$$I_{\text{rms}} = \sqrt{i_{\text{ave}}^2} \text{ and } V_{\text{rms}} = \sqrt{v_{\text{ave}}^2},$$

where

$$i_{\text{ave}}^2 = \frac{1}{T} \int_0^T i^2(t) \, dt \text{ and } v_{\text{ave}}^2 = \frac{1}{T} \int_0^T v^2(t) \, dt.$$

With $i(t) = I_0 \sin(\omega t - \phi)$ and $v(t) = V_0 \sin \omega t$, we obtain

$$I_{\text{rms}} = \frac{1}{\sqrt{2}} I_0 \text{ and } V_{\text{rms}} = \frac{1}{\sqrt{2}} V_0.$$

We may then write for the average power dissipated by a resistor,

$$P_{\text{ave}} = \frac{1}{2} I_0 V_0 = I_{\text{rms}} V_{\text{rms}} = I_{\text{rms}}^2 R. \quad (15.13)$$

This equation further emphasizes why the rms value is chosen in discussion rather than peak values. Both equations for average power are correct for **Equation 15.13**, but the rms values in the formula give a cleaner representation, so the extra factor of 1/2 is not necessary.

Alternating voltages and currents are usually described in terms of their rms values. For example, the 110 V from a household outlet is an rms value. The amplitude of this source is $110\sqrt{2} \text{ V} = 156 \text{ V}$. Because most ac meters are calibrated

in terms of rms values, a typical ac voltmeter placed across a household outlet will read 110 V.

For a capacitor and an inductor, $\phi = \pi/2$ and $-\pi/2$ rad, respectively. Since $\cos \pi/2 = \cos(-\pi/2) = 0$, we find from **Equation 15.12** that the average power dissipated by either of these elements is $P_{\text{ave}} = 0$. Capacitors and inductors absorb energy from the circuit during one half-cycle and then discharge it back to the circuit during the other half-cycle. This behavior is illustrated in the plots of **Figure 15.16**, (b) and (c), which show $p(t)$ oscillating sinusoidally about zero.

The phase angle for an ac generator may have any value. If $\cos \phi > 0$, the generator produces power; if $\cos \phi < 0$, it absorbs power. In terms of rms values, the average power of an ac generator is written as

$$P_{\text{ave}} = I_{\text{rms}} V_{\text{rms}} \cos \phi.$$

For the generator in an RLC circuit,

$$\tan \phi = \frac{X_L - X_C}{R}$$

and

$$\cos \phi = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{R}{Z}.$$

Hence the average power of the generator is

$$P_{\text{ave}} = I_{\text{rms}} V_{\text{rms}} \cos \phi = \frac{V_{\text{rms}}}{Z} V_{\text{rms}} \frac{R}{Z} = \frac{V_{\text{rms}}^2 R}{Z^2}. \quad (15.14)$$

This can also be written as

$$P_{\text{ave}} = I_{\text{rms}}^2 R,$$

which designates that the power produced by the generator is dissipated in the resistor. As we can see, Ohm's law for the rms ac is found by dividing the rms voltage by the impedance.

Example 15.3

Power Output of a Generator

An ac generator whose emf is given by

$$v(t) = (4.00 \text{ V}) \sin [(1.00 \times 10^4 \text{ rad/s})t]$$

is connected to an RLC circuit for which $L = 2.00 \times 10^{-3} \text{ H}$, $C = 4.00 \times 10^{-6} \text{ F}$, and $R = 5.00 \Omega$. (a) What is the rms voltage across the generator? (b) What is the impedance of the circuit? (c) What is the average power output of the generator?

Strategy

The rms voltage is the amplitude of the voltage times $1/\sqrt{2}$. The impedance of the circuit involves the resistance and the reactances of the capacitor and the inductor. The average power is calculated by **Equation 15.14**, or more specifically, the last part of the equation, because we have the impedance of the circuit Z , the rms voltage V_{rms} , and the resistance R .

Solution

- a. Since $V_0 = 4.00 \text{ V}$, the rms voltage across the generator is

$$V_{\text{rms}} = \frac{1}{\sqrt{2}}(4.00 \text{ V}) = 2.83 \text{ V}.$$

- b. The impedance of the circuit is

$$\begin{aligned}
 Z &= \sqrt{R^2 + (X_L - X_C)^2} \\
 &= \left\{ (5.00 \, \Omega)^2 + \left[(1.00 \times 10^4 \, \text{rad/s})(2.00 \times 10^{-3} \, \text{H}) - \frac{1}{(1.00 \times 10^4 \, \text{rad/s})(4.00 \times 10^{-6} \, \text{F})} \right]^2 \right\}^{1/2} \\
 &= 7.07 \, \Omega.
 \end{aligned}$$

c. From **Equation 15.14**, the average power transferred to the circuit is

$$P_{\text{ave}} = \frac{V_{\text{rms}}^2 R}{Z^2} = \frac{(2.83 \, \text{V})^2 (5.00 \, \Omega)}{(7.07 \, \Omega)^2} = 0.801 \, \text{W}.$$

Significance

If the resistance is much larger than the reactance of the capacitor or inductor, the average power is a dc circuit equation of $P = V^2/R$, where V replaces the rms voltage.



15.4 Check Your Understanding An ac voltmeter attached across the terminals of a 45-Hz ac generator reads 7.07 V. Write an expression for the emf of the generator.



15.5 Check Your Understanding Show that the rms voltages across a resistor, a capacitor, and an inductor in an ac circuit where the rms current is I_{rms} are given by $I_{\text{rms}}R$, $I_{\text{rms}}X_C$, and $I_{\text{rms}}X_L$, respectively. Determine these values for the components of the RLC circuit of **Equation 15.12**.

15.5 | Resonance in an AC Circuit

Learning Objectives

By the end of the section, you will be able to:

- Determine the peak ac resonant angular frequency for a RLC circuit
- Explain the width of the average power versus angular frequency curve and its significance using terms like bandwidth and quality factor

In the RLC series circuit of **Figure 15.11**, the current amplitude is, from **Equation 15.10**,

$$I_0 = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}. \quad (15.15)$$

If we can vary the frequency of the ac generator while keeping the amplitude of its output voltage constant, then the current changes accordingly. A plot of I_0 versus ω is shown in **Figure 15.17**.